

Topology of configuration spaces for indistinguishable particles on tree graphs



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Main result

We compute the homology groups of the configuration space for indistinguishable particles on tree graphs. We also discuss the difficulties that arise while trying to extend our approach to arbitrary simple graphs.

Motivation

A proper construction of a classical configuration space for indistinguishable particles (Leinaas and Myrheim '77).

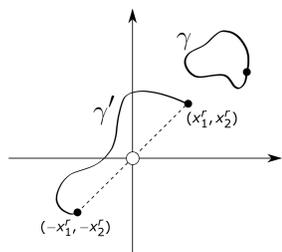
$$C_n(X) = \frac{X^{\times n} - \Delta}{S_n}$$

- consider wavefunctions on $C_n(X)$

Example - two particles in \mathbb{R}^2

$$C_2(\mathbb{R}^2) = \mathbb{R}^2 \times [(\mathbb{R}^2 - \{0\}) / \sim]$$

centre of mass relative position



- Loops, where particles exchange, are non-contractible.
- Translation of a wavefunction along a non-contractible loop can result with a multiplication by some phase factor.

$$\hat{T}_{\gamma^n} \Psi = e^{in\phi} \Psi$$

$$\hat{T}_{\gamma_1 + \gamma_2} \Psi = e^{i(\phi_1 + \phi_2)} \Psi$$

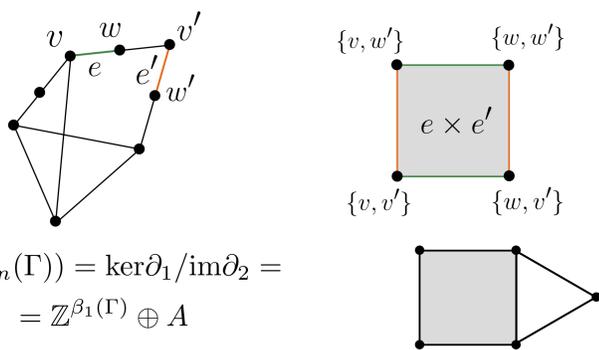
Abelian representations of $\pi_1(C_n(X))$, i.e. $H_1(C_n(X), \mathbb{Z})$.

$$H_1(C_n(\mathbb{R}^3)) = \mathbb{Z}_2, \quad H_1(C_n(\mathbb{R}^2)) = \mathbb{Z}, \quad H_1(C_n(\mathbb{R})) = 1$$

(fermions or bosons) (anyons)

Graph configuration space

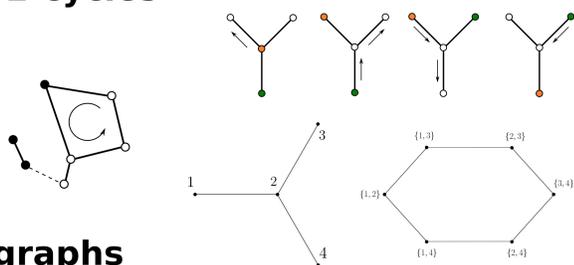
$C_n(\Gamma)$ deformation retracts to a cubic complex.



$$H_1(C_n(\Gamma)) = \ker \partial_1 / \text{im } \partial_2 = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$$

A - a quantum-statistical part, it depends on the connectivity and on the planarity of the graph.

Basis 1-cycles



Star graphs

$$\beta_1^{(n)}(S_E) = \binom{n+E-2}{E-1} (E-2) - \binom{n+E-2}{E-2} + 1$$

Higher homology groups

$$H_i(C_n(\Gamma)) = \ker \partial_i / \text{im } \partial_{i+1}$$

V.I. Arnold computed the homology groups of $C_n(\mathbb{R}^2)$. They have the following properties:

Finiteness

H_i are finite, except H_0 and $H_1, H_i(C_n(\mathbb{R}^2)) = 0, i \geq n$

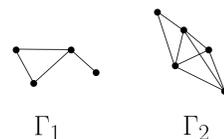
Recurrence

$$H_i(C_{2n+1}(\mathbb{R}^2)) = H_i(C_{2n}(\mathbb{R}^2))$$

Stabilisation

$$H_i(C_n(\mathbb{R}^2)) = H_i(C_{2i-2}(\mathbb{R}^2)), \quad n \geq 2i - 2$$

What about graphs?

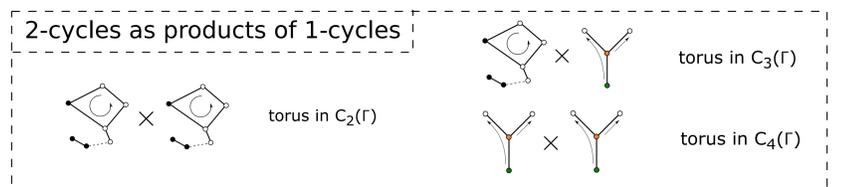


$$C_n(\Gamma_1 \sqcup \Gamma_2) = \bigsqcup_{k+l=n} C_k(\Gamma_1) \times C_l(\Gamma_2)$$

$$H_m(C_n(\Gamma_1 \sqcup \Gamma_2)) = \bigoplus_{k+l=n} H_m(C_k(\Gamma_1) \times C_l(\Gamma_2))$$

$$H_m(C_k(\Gamma_1) \times C_l(\Gamma_2)) = \bigoplus_{i+j=m} H_i(C_k(\Gamma_1)) \otimes H_j(C_l(\Gamma_2)) \quad \text{K\"unneth theorem}$$

products of cycles



Tree graphs

An example of how to handle connections between the components. Strategy:

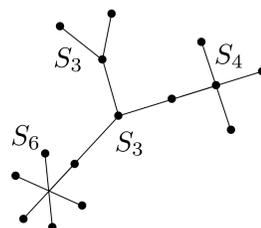
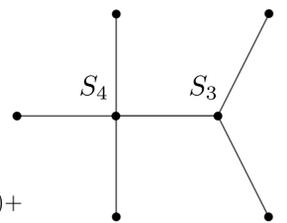
- multiply 1-cycles from the star subgraphs, i.e. from $C_k(S_3) \times C_l(S_4), k+l=n$,
- subtract the overcounted cells, stemming from the distribution of additional particles.

$$\dim H_2(C_4(T)) = \beta_1^{(2)}(S_4) \times \beta_1^{(2)}(S_3) = 3$$

$$\dim H_2(C_5(T)) = \beta_1^{(3)}(S_4) \times \beta_1^{(2)}(S_3) + \beta_1^{(2)}(S_4) \times \beta_1^{(3)}(S_3) - \beta_1^{(2)}(S_4) \times \beta_1^{(2)}(S_3) = 17$$

General procedure: regard a tree as a collection of star graphs, $S = \{S_{E_1}, S_{E_2}, \dots, S_{E_p}\}, E_k \geq 3$

$$\dim H_2(C_n(T)) = \sum_{\{S_{E_k}, S_{E_l}\} \subset S} \left(\sum_{m=2}^{n-2} \beta_1^{(m)}(S_{E_k}) \times \beta_1^{(n-m)}(S_{E_l}) + \sum_{s=2}^{n-3} \beta_1^{(s)}(S_{E_k}) \times \beta_1^{(n-s)}(S_{E_l}) \right)$$



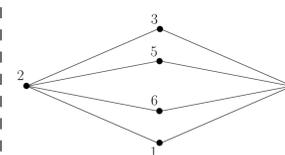
For higher homology groups, consider all m-tuples of star graphs and proceed in the same way.

- Homology groups are free and do not depend on the connections between the star graphs.
- $\dim H_m(C_n(T)) = 0$ when $n < 2m$ or $m > \#S$.
- Homology groups for particles on tree graphs do not stabilise.

General problem

What are other possible cycles?

- $C_n(\Gamma)$ are aspherical (A. Abrams, 2000).



$$H_2(C_3(\Gamma)) = \mathbb{Z}$$

$$H_1(C_3(\Gamma)) = \mathbb{Z}^6 \quad g = 3$$

$$C_3(\Gamma) \simeq \text{triple torus}$$



References

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